Alternative Modeling for Long Term Risk

Dominique Guegan, Xin Zhao

Documents de Travail du
Centre d'Economie de la Sorbonne

CES Working Papers

Alternative Modeling for Long Term Risk

Dominique GUEGAN, Xin ZHAO

2012.25

Maison des Sciences Économiques, 106-112 boulevard de L'Hôpital, 75647 Paris Cedex 13
http://centredeconomiesorbonne.univ-paris1.fr/bandeau-haut/documents-de-travail/
ISSN : 1955-611X

Alternative Modeling for Long Term Risk

Dominique Guégan[1] - Xin Zhao[2]

Abstract

In this paper, we propose an alternative approach to estimate long-term risk. Instead of using the static square root method, we use a dynamic approach based on volatility forecasting by non-linear models. We explore the possibility of improving the estimations by different models and distributions. By comparing the estimations of two risk measures, value at risk and expected shortfall, with different models and innovations at short, median and long-term horizon, we find out that the best model varies with the forecasting horizon and the generalized Pareto distribution gives the most conservative estimations with all the models at all the horizons. The empirical results show that the square root method underestimates risk at long horizon and our approach is more competitive for risk estimation at long term.

Keywords: Long Memory, Value at Risk, Expect Shortfall, Extreme Value Distribution
JEL: G32, G17, C58

1. Introduction

Long-term risk is an important but usually undervalued domain of risk management. Although there are a lot of advantages to know long-term risk, there is little speaking and attention on it. The best known approach is the square root method (J. P. Morgan (1996)[21]). Despite many criticisms and worries, this method has been widely used by practitioners and accepted by regulators[3]. However, the recent crisis, the sovereign problem and the international banking difficulties all evoke us to introspect current risk management methods. Why we were so vulnerable or less prepared when some incident trigger the problem? Were we too optimistic? In this paper, we try to answer the questions by studying the long-term risk using econometrics methodology. Concretely, we talk about which econometric model and risk measure outperforms among others from an econometric viewpoint. We will also explore whether the square root method leads to over-optimism and causes the fragile guard against the sudden arrival of crisis.

Instead of using the static approach, we choose to use a dynamic way to trace the underlying risk. We look at the dynamics of volatility. Therefore, the core problem is volatility forecasting. As soon as our main objective is long-term risk estimation, one big challenge is long-term volatility forecasting. To solve this problem, we put our hope on a widely explored phenomenon in financial assets, named long memory property in which the future observations are predictable by the observations in the past. This property has been well documented in hydrology, meteorology and geophysics (Beran (1994)[7]). In finance field, Greene and Fielitz (1977)[16] noted the existence of long-term dependence and the non-normal behavior of stock return series; Aydogan and Booth (1988)[4] re-evaluated and extended the study by using a longer data set and the results supported Greene and Fielitz's findings; Ding, Engle and Granger (1993)[12] introduced a new model by investigating the long-memory property of S&P 500 stock returns; Ding and Granger (1996)[13] extended this work on modeling volatility persistence for various speculative returns. In the risk domain, Cornelis (2005)[11] pointed that ignoring the extreme events and long memory behavior leads to the failure of VaR. Kang and Yoon (2008)[20] concluded that the VaR performance with long

[1]Corresponding author. University Paris 1 Panthéon-Sorbonne, CES UMR8174, 106 bd de l'Hopital, 75013, Paris, France, *email: dguegan@univ-paris1.fr*

[2]University Paris 1 Panthéon-Sorbonne, CES UMR8174, 106 bd de l'Hopital, 75013, Paris, France, *email: xinzhao0617@gmail.com*

[3]Basel Committee(1996): "Market Risk Amendment suggests that banks should estimate VaR for a 10-day horizon, and banks are allowed to obtain these estimates by scaling up shorter-horizon VaRs using the square root rule."

memory model outperformed others. To our knowledge, however, there is no study on the estimation of longer horizon for VaR for more than several days using long memory methodology. We believe it is interesting to extend the previous works to risk estimation at longer time span. Then we study short, median, and long-term risk considering short and long-memory models. We find that the long-memory model outperforms the others at long term span.

Another interesting point in econometric modeling is the choice of the distribution for innovations. Most of the volatility models use Gaussian distribution, thus fail to capture the extreme events. This simple assumption generates error estimations for long-term risk management. Corneils (2005)[11] discussed this problem. Kang and Yoon (2008)[20] also proved the skewed t-Student distribution outperformed the widely used Gaussian distribution. We expect the extreme value distributions, such as Generalized Pareto Distribution (GPD), can improve the estimation of VaR (Guégan and Hassani (2011)[19]). Therefore, we compare the estimations of each model with different distributions. Four distributions coming from three probability distribution families are considered in this paper, and the empirical results show that GPD improve the estimation of long term risks with all the models.

Since our object is risk measure, the criterion of the best option is decided by the performance of the risk measures which are obtained by the underlying model and distribution. The two risk measures we discussed in this paper are Value-at-Risk (Alexander, Frey and Embrechts (2005)[2]) and Expected Shortfall (Rockafellar and Uryasev(2000)[23]). The two risk measures are preferred by practitioners and recommended by regulators. They are also flexible with different models and distributions. To calculate the long-term VaR and ES, we follow the conventional notions. The difference between our approach and the square root method is that we compute the long-term VaR and ES using the aggregated losses simulated by the underlying model and distribution rather than simply scaling the one-day estimations to longer-term. By our method, we can capture the future dynamics of financial series. The empirical results support the fact that our methodology beats the square root method for long-term risk estimation on the basis of the results of the exceedance ratio test. Indeed, the square root method underestimates long-term risk. Long-memory models and extreme value distributions improve the results.

In the next section, we introduce the modeling and the two risk measures. Data are analyzed in section 3. The in-sample and out-of-sample analysis are exhibited in section 4 and 5. Remarks and conclusions of our study are given in section 6.

2. Modeling

We take two steps to obtain the risk measures at the given horizon. Firstly, we model the interested financial series, stock returns. Secondly, we compute the two risk measures, VaR and ES, using the aggregated returns (or losses) computed by the model in step one. We firstly introduce the modeling of returns, then the algorithm for risk measures at given horizon.

2.1. Modeling of Returns

The financial series studied in this paper are daily log-returns of equities, denoted $\{r_1, r_2, \dots, r_n\}$. To model the series, we propose a general model which captures most of the properties of financial series: the k-factor GIGARCH Model (Guégan (2000[17], 2003[18]))

$$\begin{cases} \Phi_L(B)\ \Pi_{i=1}^{k}(I - 2\nu_{L,i}B + B^2)^{d_{L,i}} r_t = \Theta_L(B)\ \varepsilon_t \\ \varepsilon_t = h_t \epsilon_t \\ \Phi_V(B)\ \Pi_{j=1}^{\kappa}(I - 2\nu_{V,j}B + B^2)^{d_{V,j}} h_t^{\delta} = \Theta_V(B)\ \epsilon_t^{\delta}, \end{cases} \tag{1}$$

where r_t are the observed returns, ϵ_t are i.i.d. innovations which follow a given probability distribution, $d_{L,i}$ and $d_{V,j}$ are the long memory parameters for level and volatility, $\nu_{L,i}$ and $\nu_{V,j}$ are the frequency location parameters for level and volatility, $\Phi_L(B)$, $\Phi_V(B)$, $\Psi_L(B)$ and $\Psi_V(B)$ are ARMA operators, and B is the backward difference operator.

The k-factor GIGARCH Model (equation (1)) takes the long-memory property of financial series into account through parameters $d_{L,i}$ and $d_{V,j}$; it models the short-memory behavior with ARMA operators $\Phi_L(B)$, $\Phi_V(B)$, $\Psi_L(B)$ and $\Psi_V(B)$; the seasonality phenomenon with parameters $\nu_{L,i}$ and $\nu_{V,j}$; the asymmetric and fat-tail features by choosing a proper distribution of innovation ϵ_t. In the empirical study, we use the following models which are particular case of the previous one:

$$\text{AR}: \quad r_t = \sum_{k=1}^{p} \phi_k r_{t-k} + \varepsilon_t, \quad \varepsilon_t \sim i.i.d. \tag{2}$$

$$\begin{aligned} \text{GARCH}: \quad & r_t = u + \varepsilon_t, \\ & \varepsilon_t = h_t \epsilon_t, \quad \epsilon_t \sim i.i.d. \\ & h_t^2 = \alpha_0 + \sum_{i=1}^{P} \alpha_i h_{t-i}^2 + \sum_{j=1}^{Q} \beta_j \varepsilon_{t-j}^2 \\ & \sum_{i=1}^{P} \alpha_i + \sum_{j=1}^{Q} \beta_j < 1 \end{aligned} \tag{3}$$

$$\begin{aligned} \text{IGARCH}: \quad & r_t = u + \varepsilon_t, \\ & \varepsilon_t = h_t \epsilon_t, \quad \epsilon_t \sim i.i.d. \\ & h_t^2 = \alpha_0 + \sum_{i=1}^{P} \alpha_i h_{t-i}^2 + \sum_{j=1}^{Q} \beta_j \varepsilon_{t-j}^2 \\ & \sum_{i=1}^{P} \alpha_i + \sum_{j=1}^{Q} \beta_j = 1 \end{aligned} \tag{4}$$

$$\begin{aligned} \text{FIGARCH}: \quad & r_t = u + \varepsilon_t, \\ & \sum_{j=0}^{\infty} \frac{\Gamma(j-d)}{\Gamma(j+1)\Gamma(-d)} B^j \varepsilon_t^2 = \eta_t = h_t^2 \epsilon_t^2, \quad \epsilon_t \sim i.i.d. \\ & h_t^2 = \alpha_0 + \sum_{i=1}^{P} \alpha_i h_{t-i}^2 + \sum_{j=1}^{Q} \beta_j \eta_{t-j} \\ & \sum_{i=1}^{P} \alpha_i + \sum_{j=1}^{Q} \beta_j < 1 \end{aligned} \tag{5}$$

The parameters p, P and Q are the number of the orders estimated using the AIC (Akaike (1974)[1]) and BIC values (Schwarz (1978)[24]); n is the sample size; u is the mean of returns; d is the long-memory parameter for volatility; $\phi_1, \phi_2, \ldots, \phi_p$; $\alpha_0, \alpha_1, \ldots, \alpha_P$; $\beta_1, \beta_2, \ldots, \beta_Q$ are real numbers. The *i.i.d* innovations in equation (2) - (5) are usually assumed as Gaussian distribution. In this study we consider also Student distribution (Kang and Yoon (2008)[20]), Normal Inverse Gaussian (NIG) distribution (Barndorff-Nielsen (1997)[6]), and Generalized Pareto Distribution (GPD) (Embrechts *et al* (1997)[14]). Assuming that we observe the returns $\{r_1, \ldots, r_n\}$, we firstly estimate the long memory parameter d using the Whittle method (Palma (2007)[22]), then we estimate the other parameters using the recursive estimation method (Chorro *et al.* (2010)[10]).

2.2. Risk Measures

We describe now the two risk measures considered in this paper and explain how we compute them for the long-term. VaR of the portfolio during a period of T days at the given confidence level $\alpha \in (0,1)$

is equal to:

$$VaR_\alpha(X_T) = \inf\{x \in \mathbf{R} : P(X_T > x) \leq 1 - \alpha\}$$

Here X_T denote the loss of the portfolio during T days since day t, which is obtained by summing the T days returns, $\{r_{t+1}, \ldots, r_{t+T}\}$, $X_T = \sum_{i=1}^{T} r_{t+i}$, where, $T \in \mathbb{Z}$, $r_t = \log \frac{P_t}{P_{t-1}}$. P_t are the observed close price on the market. We simulate N times the series $\{r_{t+1}, \ldots, r_{t+T}\}$ in order to obtain a simulated sample of the T days loss of the portfolio, $\{X_{T,1}, \ldots, X_{T,N}\}$. The VaR at time T associated to this portfolio comes out to be the $1 - \alpha$ quantile of the empirical distribution. Likewise, to get a risk measure at time $T + h$, we follow the same procedure, simulating N processes of the series $\{r_1, \ldots, r_{T+h}\}$ to get the empirical distribution of $T + h$ days' return.

The second risk measure is the expected shortfall, named also conditional value at risk (CVaR), average value at risk (AVaR), and expected tail loss (ETL). It is an alternative risk measure to VaR which is considered as a more conservative risk measure and is a coherent risk measure (Artzner *et al* (1999)[3]). It is equal to $E(X|X > VaR_\alpha)$. Given the empirical distribution of T days return of the portfolio, $\{X_{T,1}, \ldots, X_{T,N}\}$, the ES corresponds to the loss which is below the estimated VaR, where VaR_α is estimated in previous step:

$$\widehat{ES}_\alpha = Mean(X_i | X_i > VaR_\alpha)$$

3. Data

The data used in this paper are the daily returns of the Standard and Poor's 500 Composite Index from 1 January 1990 to 31 December 2010. We take the first 16 years' data, from 1900 to 2005, as the in-sample data and the last 5 years' data, from 2006 to 2010, as the frame of reference to check our out-of-sample estimations (Figure 1). There are totally 4715 in-sample daily data and 1304 out-of-sample daily data.

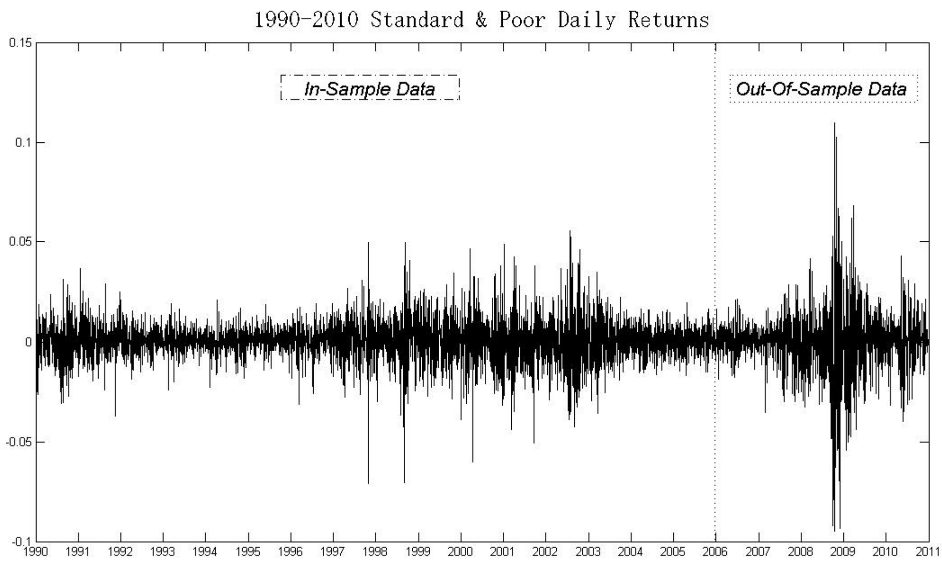

Figure 1: In-Sample and Out-of-Sample Data (1990-2010)

3.1. Data Analysis

The returns we used in our study are the logarithmic returns converted by the daily close price: $r_t = \ln(\frac{P_t}{P_{t-1}})$. The in-sample process covers a relative long time span, which contains the currency crisis in European Exchange Rate around 1992-1993 and the Asia crisis from 1997 to 1998. The out-of-sample data span coincides with the recent crisis and we can see the high volatility during these turbulent periods from Figure 1. We analyze the performance of the two risk measures on this tumultuous period. To start our analysis, we provide the statistics and tests for the in-sample data series in Table 1.

Statistics	Value
Mean	$3.0226 \cdot 10^{-4}$
Median	$1.0634 \cdot 10^{-4}$
Standard Deviation	0.0100
Skewness	-0.1005
Kurtosis	7.0124

Test	P-Value
Variance Ratio Test	$3.7826 \cdot 10^{-68}$
ADF[4] Test	$1.0000 \cdot 10^{-3}$
ARCH Test	0
Whittle Test on Return	0.3462
Whittle Test on Volatility	$4.7022 \cdot 10^{-12}$

Table 1: Statistics and Tests of In-Sample Data (1990 - 2005)

Firstly, from the P-value of the variance ratio test and the statistics, we observe that the process is neither a white noise nor follows the Gaussian distribution. It means that the assumptions underlying the square-root method is not satisfied. Secondly, the P-value of ADF Test indicates that the data appear stationary. Thirdly, the P-value of ARCH test indicates that the volatility is not constant. Finally, the P-values of the Whittle tests show that there is no long-memory behavior inside the $\{r_t\}_{t=1}^n$ process but there is significant long-memory behavior for the $\{r_t^2\}_{t=1}^n$ process.

3.2. Parameter Estimation

Based on the previous remarks, we now use the four models introduced in equations (2)-(5) with four distributions, the Normal, Student, NIG and GPD distributions, to model the returns and their volatilities, and to compute the two risk measures. To setup our study, we need firstly estimate the parameters of the sixteen models. The long-memory parameter d in volatility is estimated using the squared returns $\{r_1^2, r_2^2, \ldots, r_n^2\}$ by the Whittle method (Palma (2007)[22]), and $\hat{d} = 0.1758$. The other parameters are estimated by the Recursive Estimation method proposed by Chorro *el al.* (2010)[10]. The estimations are given in Table B.3 - B.6.

4. In-Sample Results

In this section, we use the algorithm introduced in section 2 and parameters estimated in section 3 to simulate the distribution of aggregated returns in the short, median and long-term, and then compute the values of VaR and ES measures.

Firstly, we specify the notions of the short, median and long-term in our study. Intuitively, these notions should not be the same for all financial markets. They depend on the liquidity of the market. For stock market, where investment can change everyday, half a year is a long-term. Thus, we define the period less than 1 month as the short-term, 1 month to 3 months as the median-term, 6 months and longer as the long-term.

Accordingly, we compute and compare the estimations of VaR and ES by the models introduced in equations (2)-(5) and the Normal, Student, NIG and GPD distributions on the monthly, quarterly, and half a yearly horizons. Moreover, we compare our estimations with the square-root estimations using the criteria of exceedance ratio[5].

4.1. Short-Term Estimation

We simulate the distribution of the future 30 days' returns from the 3rd January 1990 to the 21th November 2005. There are 4143 monthly returns[6], and correspondingly we provide 4143 estimations of

[5] The ratio is equal to the number of the losses that are bigger than the estimated VaR over the size of the estimations

[6] We compute the 30 days' log-returns by the raw data.

VaR and 4143 estimations of ES which are computed by the simulated distributions. The estimations of VaR against the observed monthly returns are plotted in Figure 2.

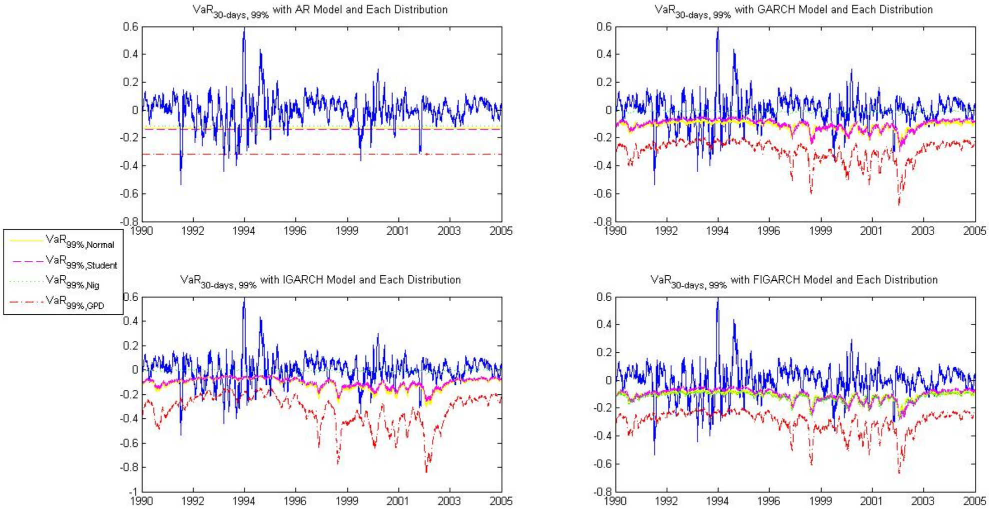

Figure 2: In-Sample (1990-2005) $VaR_{30-days,99\%}$ with Different Models and Innovations

Figure 2 shows that the VaR estimated using the linear model fails to adjust correctly to the behavior of real data. The VaR estimations by the simulated distributions with the three non-linear models, however, provide results close to the changes of volatility. To make it clear and easy to compare the results of the three non-linear models, we plot the exceedance ratios in Figure 3, where the exceedance ratio is equal to the number of losses which are bigger than the corresponding VaRs over the size of the data set.

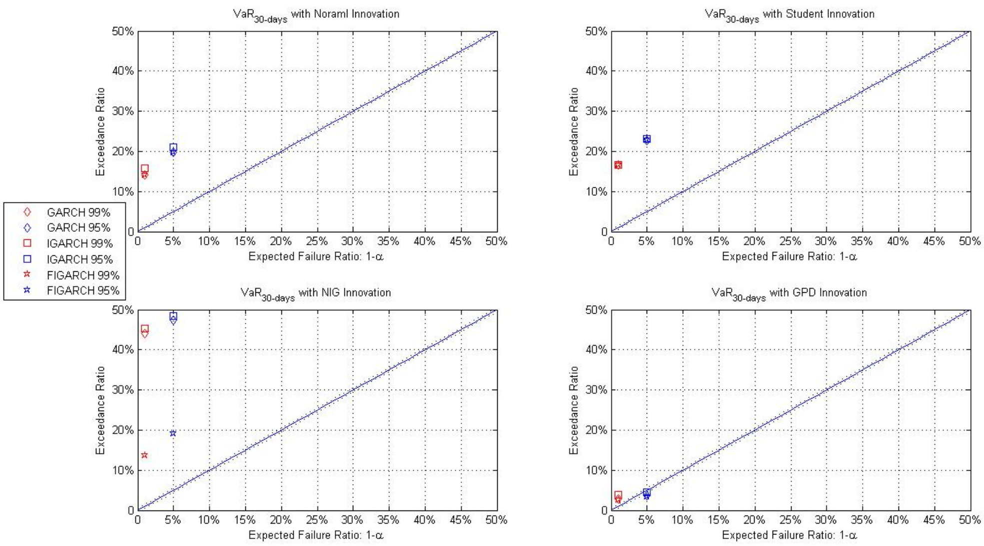

Figure 3: Exceedance Ratio of In-Sample (1990-2005) 1 Month VaR

The x-axis of each sub-figure in Figure 3 is the value of $1-\alpha$, where α is the given confidence level of the risk measure VaR. The y-axis is the value of exceedance ratios computed by comparing the estimated

VaR and the true monthly returns. Therefore, a good estimation of VaR should locate at the 45 degree line, which means the true failure ratio of VaR is the same as the expected ratio $1-\alpha$. VaR locates above the line means underestimation, below means overestimation. Applying these criteria on Figure 3, Figure A.5 and Figure A.6, we come to the following conclusions:

1. GPD distribution gives the best estimations[7].
2. FIGARCH model gives more conservative VaR than GARCH and IGARCH model[8].
3. Square root method underestimates VaR with all the distributions and models[9].

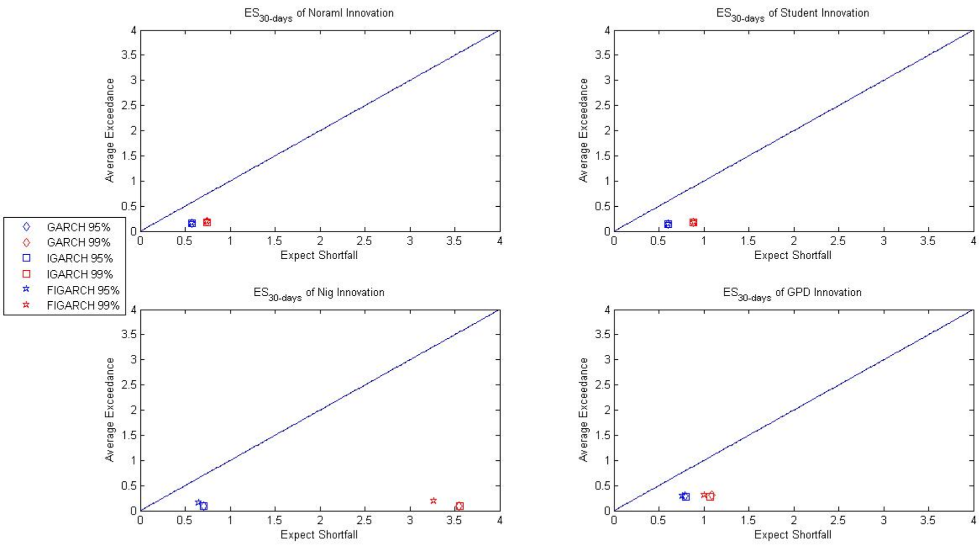

Figure 4: In-Sample (1990-2005) 1-Month ES vs. Average Loss over Corresponding VaR

Figure 4 plots the estimations of $ES_{1-month}$ against the average loss over the corresponding $VaR_{1-month}$. Similarly, the best estimation locates on the 45 degree line, because it means the true average loss over the threshold (the estimation of $VaR_{1-month}$) is equal to the expect average loss over the threshold. Points settled above (below) the 45 degree line are underestimated (overestimated) ES. The square root ES are displayed in Figure A.7. By Figure 4 and Figure A.7, we have the following deductions:

1. The dynamic approach leads to overestimation[10].
2. The GPD distribution with the dynamic approach gives the best estimation[11].
3. The FIGARCH model with the dynamic approach gives the best estimaiton[12].
4. The square root method leads to underestimations[13].

4.2. Median-Term Estimation

We do the same exercise for the 3-months horizon. There are 4083 quarterly returns, and correspondingly 4083 estimations of VaR and 4083 estimations of ES. Figure A.8 shows the exceedance ratio of the

[7] From figure 3, we observe the estimations with GPD innovations are much closer to the 45 degree line than with the other innovations. This result indicates that the aggregate losses simulated by the Normal, Student and NIG distribution fail to capture the 'true' downside possibility of aggregate returns in 1-month.

[8] FigureA.5 displays a zoomed version of the exceedance ratio with GPD distribution. Comparing the exceedance ratios inside each sub-figure of figure 3. We obtain that the biggest value of the risk estimations are given by FIGARCH model

[9] Figure A.6 shows the exceedance ratio with each innovation and model using square root method.

[10] All the points in Figure 4 are below the 45 degree line.

[11] Among the four subfigures in Figure 4, the points in the figure "$ES_{30-days}$ of GPD Innovation" are closer to the 45 degree line than the points in the other figures.

[12] In each subfigure in Figure 4, the estimation with FIGARCH model is the point closest to the 45 degree line.

[13] All the points in Figure A.7 are above the 45 degree line.

3-months VaR and the square root VaR. We observe the similar features as in the short-term analysis[14]. While, the estimation of VaR with FIGARCH model and GPD distribution has the exceedance ratio of 0.9% at the 95% confidence ($1 - \alpha = 5\%$). The estimation is good but could be too conservative for practitioners. Besides, we plot the 3-months ES by the dynamic approach and the square root method in Figure A.10. The pattern of the diagrams is close to the 1-month estimations[15].

4.3. Long-Term Estimation

For 6-month analysis, we have 4053 estimations of VaR and 4053 estimations of ES from the 3 January 1990 to 14 July 2005. The plots of the exceedance ratios in Figure A.11 confirm the findings that we observed in the short-term and median-term analysis.

The trend is more significant at the longer horizon. Firstly, the exceedance ratio of $VaR_{6-months}$ with NIG distribution at 95% confidence level are bigger than 50% with GARCH and IGARCH model. That is to say, if we use this innovation to estimate the half year VaR during the fifteen years, more than half chance, the VaR will fail. Secondly, although all the estimations of VaRs with GPD innovation largely improve the performance of the estimations of VaRs with the other distributions, the exceedance ratios are smaller than 0.5% at 95% and 99% confidence level (Figure A.12) indicating they are too conservative estimations. Figure A.13 gives the estimation of $ES_{6-months}$ and the average loss over the corresponding $VaR_{6-months}$. We observe the overestimations by the dynamic approach. But the square root ES with GPD innovations in Figure A.13 are close to the average loss over the corresponding $VaR_{6-months}$, which could be interesting for the practitioners.

Briefly, in this section, we compare the estimations of the two risk measures, VaR and ES, by the three dynamic models with the four distributions at the short-term, median-term and long-term horizons. The square root method causes underestimations of VaRs under all conditions. It gets worse on the longer time span. GPD distribution with the dynamic approach can improve the estimations of VaRs at all horizons with all models, but it becomes very conservative for long-term risk estimation. Moreover, FIGARCH model gives more conservative VaRs than the other two models. This is more significant at long-term. Furthermore, the square root method underestimate ES at the short-term and median-term horizon, but gives better estimations of ES at long-term in practice viewpoint. The dynamical ES are bigger than the 'true' average loss over VaR, but they are too conservative at the long-term horizon. The results for longer term, such as 1-year, 3-year, and 5-year are the same, so we do not repeat the analysis.

5. Out-of-Sample Results

In this section, using the same methodology as before, we estimate the monthly, quarterly, and semi-annually VaR and ES. Assuming stationarity in the process, we use the previous simulated models. There are two differences comparing to the in-sample analysis. First, we do not move forward the start day of the estimations. Here, we only have one estimation of VaR and ES at a given horizon. It corresponds to the risk measures starting from the 1st January of 2006 to a given future time. Second, we forecast volatility to simulate the future returns.

5.1. Short-Term Forecasting

The estimatons of the 1-month VaR and ES are listed in Table 2. To evaluate the results, we compare the value of the estimated VaR and ES with the observed losses and verify whether the out-of-sample estimations have the same futures of the in-sample results.

[14]Except for GPD distribution, all the models with the other distributions underestimate the possibility of loss at 3-months horizon. Among the three non-linear model, FIGARCH gives the most conservative VaR. Moreover, the square root method underestimates the 3-months VaR under all the conditions.

[15]The dynamic approach lead to overestimation, but the square root method lead to underestimations. The GPD distribution with the dynamic approach gives the best estimation and FIGARCH model with the dynamic approach gives the best estimaiton.

(a) 1-Month Ahead VaR Estimation

Model	α	Normal	Student	NIG	GPD
GARCH(1,1)	0.95	0.0452	0.0415	0.0713	0.2325
	0.99	0.0784	0.0753	0.1010	0.2879
IGARCH(1,1,1)	0.95	0.0434	0.0519	0.0811	0.2625
	0.99	0.0674	0.0843	0.1211	0.3858
FIGARCH(1,d,1)	0.95	0.0479	0.0492	0.0779	0.2043
	0.99	0.0708	0.0735	0.1188	0.3317

(b) 1-Month Ahead ES Estimation

Model	α	Normal	Student	NIG	GPD
GARCH(1,1)	0.95	0.0658	0.0824	0.1058	0.2788
	0.99	0.1166	0.0945	0.1419	0.3013
IGARCH(1,1,1)	0.95	0.0777	0.0857	0.1057	0.2625
	0.99	0.1054	0.1757	0.1423	0.3858
FIGARCH(1,d,1)	0.95	0.0762	0.0842	0.1102	0.2884
	0.99	0.1063	0.0984	0.1339	0.3811

Table 2: Out-of-Sample (2006-2010) 1-Month Ahead VaR & ES Estimation

The first monthly return of 2006 is positive. Since all the estimations in Table 2 are bigger than the real loss, there is no exceedance over the estimations. We conclude the short-term forecasting results in Figure 2 as follows:

1. The GPD innovation gives the biggest VaR and ES with all models.
2. IGARCH model gives more accurate volatility forecasting in the short-term[16].
3. As we expected, ES is a more conservative risk measure than VaR[17].

Besides, the estimations with Normal, Student and NIG distribution are closed to the squared root results, but the simulated VaR and ES with the GPD distribution are bigger than the square root VaR and ES with GPD distribution. This is also consistent with the in-sample results.

5.2. Median-Term Forecasting

We provide the estimations of the 3-months VaR and ES in Table B.7. The results are similar as the short-term estimations. Firstly, since the real return of the first quarter of 2006 is positive, there is no exceedance of all estimations. Secondly, GPD is still the distribution that gives the biggest value of VaR and ES with all models. The difference between the estimations with GPD and the other three distributions are much larger than the difference at the 1-month horizon. All the ES are bigger than the corresponding VaR. The mean square errors of the 3-month volatility forecasting in Table B.10 reveal that the IGARCH model gives the most accurate forecasts among the three non-linear models with all distributions at the 3-month horizon. Moreover, the differences between the square root results and the simulated results are the same as before.

5.3. Long-Term Forecasting

The characteristics observed in the short and median-term results are also exhibited with the 6-months' estimations in Table B.8. The values of the two risk measures by GPD distribution are bigger than the value estimated by the other distributions. This fact coincides with the in-sample results. The longer the forecasting horizon, the more conservative estimations obtained by GPD innovation. Moreover, the mean square errors of the volatility forecasting in Table B.10 at 6-months horizon show the IGARCH model is still the best option for volatility forecasting. But IGARCH model is a non-stationary model and has

[16] The mean square errors of volatility forecasting in Table B.10 indicate that the 'best' volatility forecasting at 1-month horizon is given by IGARCH model with all the distributions.

[17] The values of ES are bigger than the values of the corresponding VaR.

no forecast ability in the long-term. The same remark works for the stationary short-memory GARCH models. Table B.10 shows that up from one year horizon, the forecast ability of FIGARCH outperforms the other models. This fact indicates that the FIGARCH model is a better choice for the assets in the less liquidity market with the 'long-term' when it is longer than one year. Besides, we perceive the same features by comparing the results with simulated approach and the square root approach.

Moreover, since the returns during the first 6 month of 2006 are mainly positive, it is difficult to distinguish the performance of the estimations using the monthly, quarterly and half yearly returns during this period (the 1st Jan. 2006 to 30th June 2006). The conservative risk measures such as the results with GPD innovation appear to be less interesting, but it is important to check the performance of the estimations during the period with big losses. In the out-of-sample data set, the biggest loss[18] occurs at the 9th March 2009 with an aggregate loss of 0.6126. The estimations of the two risk measures with the forecasting horizon around 3 years, from the 1st January 2006 to the 9th March 2009, are given in Table B.9. The results show that only the risk estimations with the GPD innovation can capture this extreme loss at the 3 years horizon, which means the estimated values are bigger than the aggregate loss 0.6126[19]. Besides, the FIGARCH model has more precise volatility forecasting and turns out to be a more interesting model for long-term estimations.

6. Conclusion

In summary, this empirical study gives us several interesting results. We comment them from the standpoint of risk measurement or risk management:

- First, the in-sample tests show that the traditional distributions applied in risk measurement fail to give good measures in short, median and long-term horizon. Gaussian innovation cannot even give good estimations for the in-sample daily risk measures. The estimations of the out-of-sample extreme loss (Table B.9) show that only the extreme value distribution GPD can capture the extreme loss. All these results suggest that risk managers should leave the idealistic assumption of Gaussian innovation and is more realistic to use the more conservative assumptions in their analysis, especially for the long-term risk estimations.

- Second, the in-sample results disclose that FIGARCH model gives more conservative risk measures than the other two models with all distributions. Out-of-sample test shows that with FIGARCH model we have more accurate volatility forecasting in the long range (Table B.10), whereas, the IGARCH model performs better during the horizon up to 6-month. These results imply that there is predictability of volatility with FIGARCH model when there is long-memory behavior in the volatility, and it is a better option for risk estimation at relative long time span. Moreover, the results also indicate that we should choose different models at different time span to improve the accuracy of the risk measures.

- Finally, we compare all the estimations using dynamic approaches with the estimations using the square root method. We detect the significant underestimation of the squared risk measures at all horizons. Even with the GPD distribution, the square root method yields significant underestimation for long-term risk. The results caution us the danger of applying the square root method without verifying the restrictive assumptions of the method.

In conclusion, we propose to use different non-linear models with Generalized Pareto distribution to estimate risks in different horizons. This approach emphasizes on the dynamics of the volatility. The empirical results show that the traditional distributions and the square root method underestimate the risks

[18]The biggest loss is the smallest aggregate returns start from 1st January 2006 until the end of 2010.

[19]The results show that all the estimations with Normal, Student and NIG distribution fail to catch this extreme loss, even the more conservative risk measure ES. Whereas, the estimations with GPD innovation are able to capture this extreme loss.

at all horizons. The GPD innovation improves the estimations at the short-term, provides good estimations at the median-term, and the most conservative estimations at the long-term. The FIGARCH model has the most conservative in-sample estimations and most accurate out-of-sample volatility forecasting in the long range. The FIGARCH model with GPD distribution can capture the extreme out-of-sample losses, and give better risk measures than the traditional approach.

Reference

[1] Akaike H. (1974) "A new look at the statistical model identification". *IEEE Transactions on Automatic Control*, 19(6):716-723.

[2] Alexander M., Frey R. and Embrechts P. (2005) "Quantitative Risk Management: Concepts Techniques and Tools". *Princeton University Press, Princeton and Oxford.*

[3] Artzner P., Freddy D., Eber J. M. and Heath D. (1999) "Coherent Measures of Risk", *Mathematical Finance*, 9(3):203-228.

[4] Aydogan K. and Booth G. C. (1988) "Are there long cycles in common stock returns?" *Southern Economic Journal*, 55:141-149.

[5] Baillie R. T., Bollerslev T. and Mikkelsen H. O. (1996) "Fractionally Integrated Generalized Autoregressive Conditional Heteroskedasticity", *Journal of Econometrics*, 74:3-30.

[6] Barndorff-Nielsen O. E. (1997) "Normal inverse Gaussian disributions and stochastic volatility modeling", *Scand. J. Statist.*, 24:1-13.

[7] Beran, J. (1994) "Statistics for Long-Memory Processes", *Chapman & Hall, London.*

[8] Bollerslev T. (1986) "Generalized Autoregressive Conditional Heteroskedasticity", *Journal of Econometrics.* 31:307-327.

[9] Box G. E. P. and Jenkins G. (1976) "Time Series Analysis: Forecasting and Control", *John Wiley & Sons, New York, Chichester, Brisbane and Toronto.*

[10] Chorro C., Guégan D., and Ielop F. (2010) "Likelihood-Related Estimation Methods and Non-Gaussian GARCH Processes", *WP-2010-67, Université Paris 1, Panthéon-Sorbonne, Paris, France.*

[11] Corneils L. A. (2005) "Why VaR Fails: Long Memory and Extreme Events in Financial Markets", *Journal of financial Economics*, 3(3):19-36.

[12] Ding Z., Granger C. W. J., and Engle R. F. (1993) "A Long Memory Property of Stock Market Returns and a New Model", *Journal of Empirical Finance*, 1:83-106.

[13] Ding Z. and Granger C. W. J. (1996) "Modeling volatility persistence of speculative returns: A new approach", *Journal of Econometrics*, 73:185-215.

[14] Embrechts P., Kluppelberg C. and Mikosch T. (1997) "Modeling extremal events for insurance", *Springer Verlag, Berlin.*

[15] Granger C. W. J. (1980) "Long Memory Relationships and the Aggregation of Dynamic Models.", *Journal of Econometrics*, 14:227-238.

[16] Greene M. and Fielitz B. (1977) "Long-Term Dependence in Common Stock Returns", *Journal of Financial Economics*, 4:339-349.

[17] Guégan D. (2000) "A new model: the k-factor GIGARCH Process", *Journal of Signal Processing*, 4:265-271.

[18] Guégan D. (2003) "A Prospective study of the k-factor Gegenbauer process with heteroscedastic errors and an application of inflation rates", *Finance India*, 17:1-20.

[19] Guégan, D. and Hassani, B. K. (2011) "Operational risk: A basel ii++ step before basel iii", *WP-2011-53, Université Paris 1, Panthéon-Sorbonne, Paris, France.*

[20] Kang S. Y. and Yoon S. M. (2008) "Value-at-Risk Analysis of the Long Memory Volatility Process: The Case of Individual Stock Returns", *The Korean Journal of Finance*, 21:4-18.

[21] J. P. Morgan (1996) "RiskMetrics - Technical Document", *Fourth Edition, New York.*

[22] Palma W. (2007) "Long-Memory Time Series", *Wiley Series in Probability and Statistics, Hoboken, New Jersey.*

[23] Rockafellar R. T. and Uryasev S. (2000), "Optimization of Conditional Value-At-Risk", *The Journal of Risk*, 2(3):21-41.

[24] Schwarz, G. E. (1978). "Estimating the dimension of a model". *Annals of Statistics*, 6(2):461- 464.

Appendix A. Figures

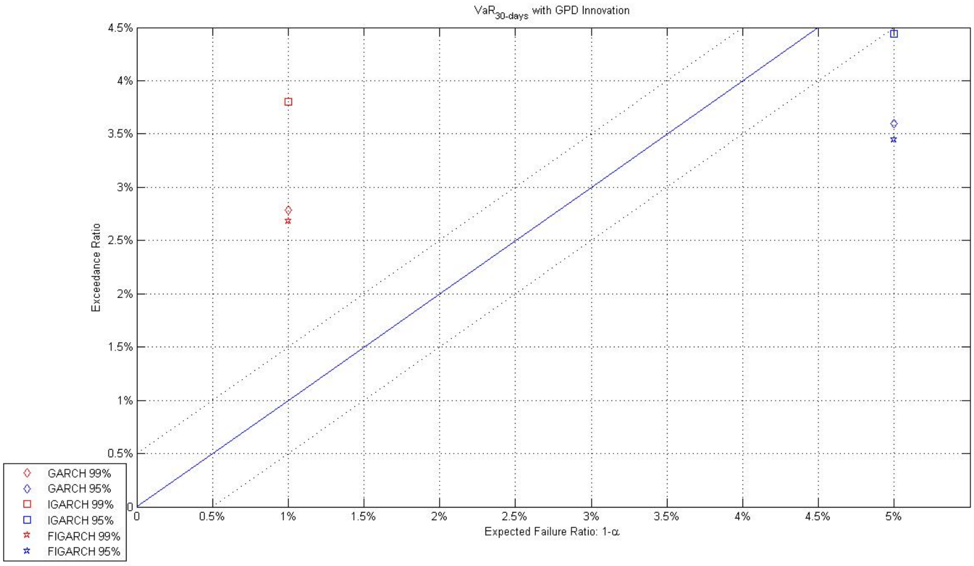

Figure A.5: Exceedance Ratio of In-Sample (1990-2005) 1-Month VaR with GPD

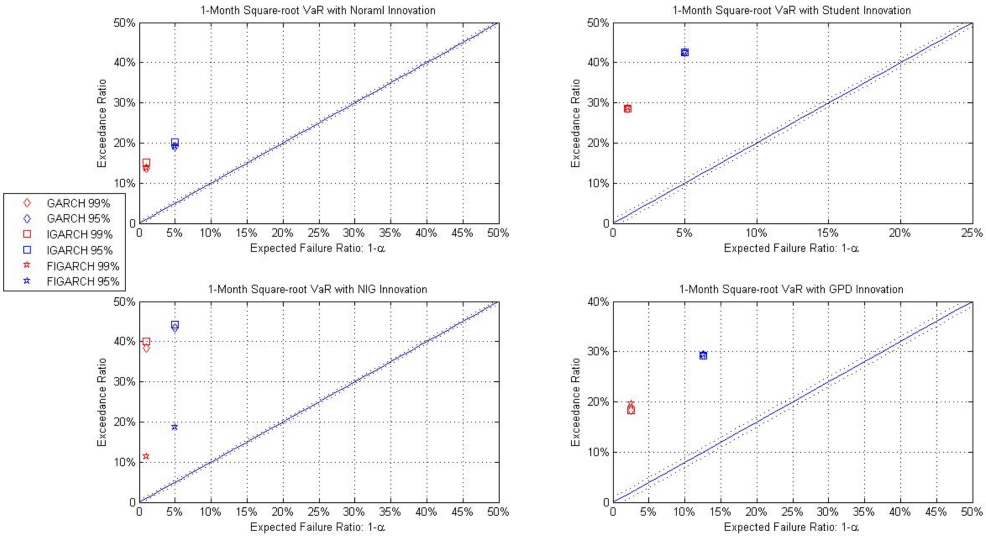

Figure A.6: Exceedance Ratio of In-Sample (1990-2005) 1-Month Square Root VaR

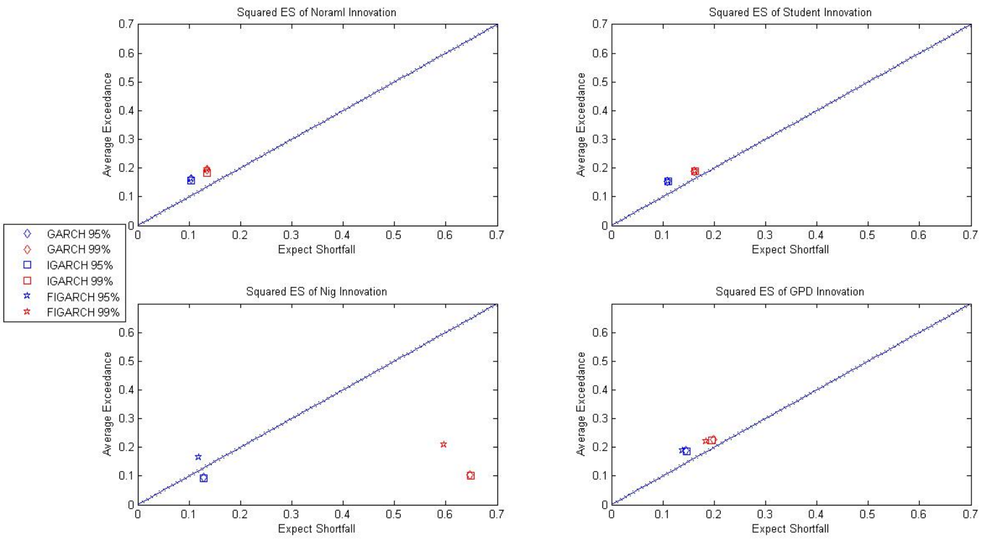

Figure A.7: In-Sample (1990-2005) Square-root estimated 1-Month ES vs. Average Observed 1-Month Losses over Corresponding VaR

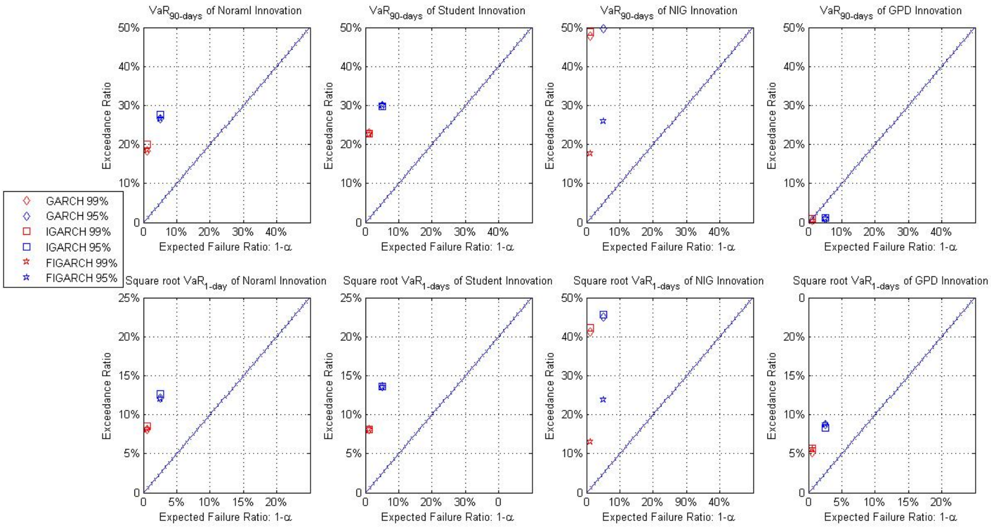

Figure A.8: Exceedance Ratio of In-Sample (1990-2005) 3-Month VaR & Square Rooted 3-Month VaR

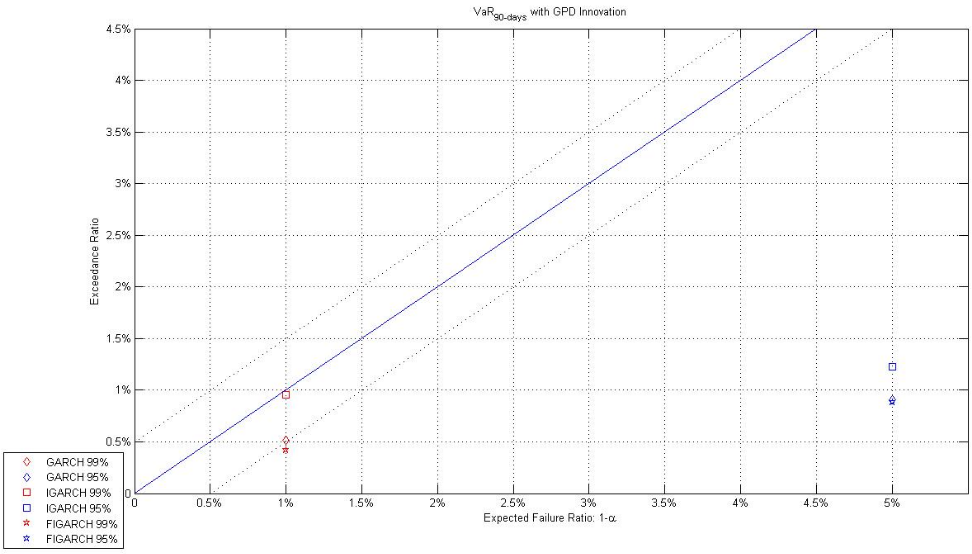

Figure A.9: Exceedance Ratio of In-Sample (1990-2005) 3-Months VaR with GPD Innovation

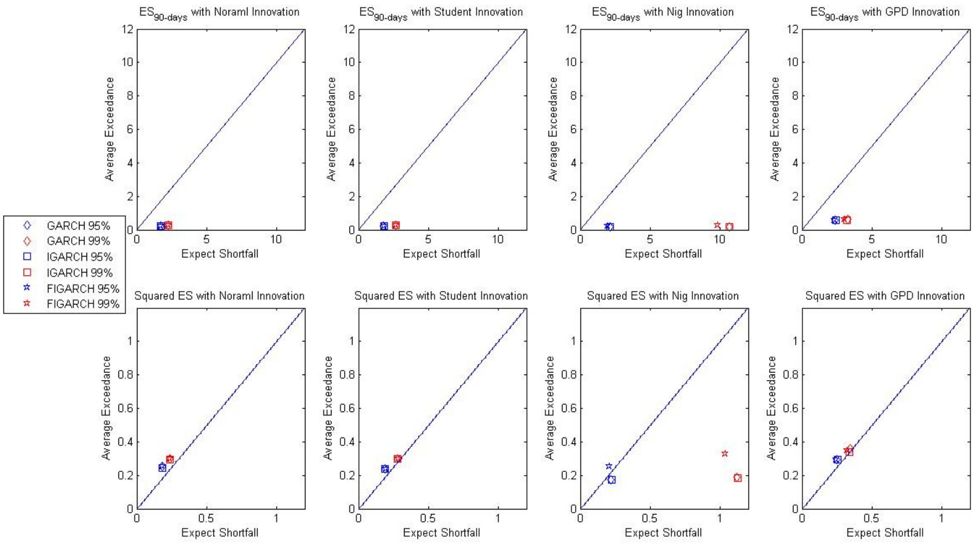

Figure A.10: In-Sample (1990-2005) $ES_{3-months}$ & Square-root estimated 3-Months ES vs. Average Observed 3 months Losses over corresponding VaR

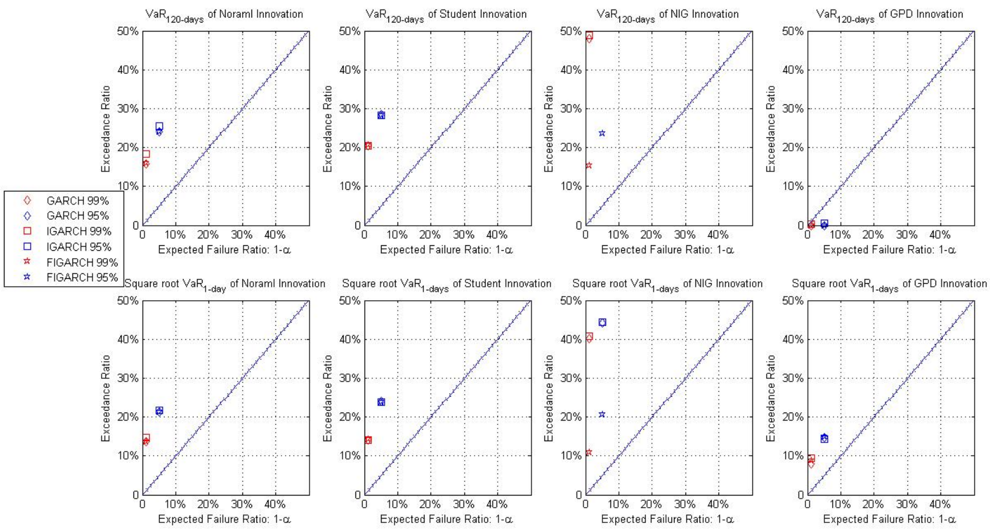

Figure A.11: Exceedance Ratio of In-Sample (1990-2005) 6-Months VaR & Square Root Estimated VaR

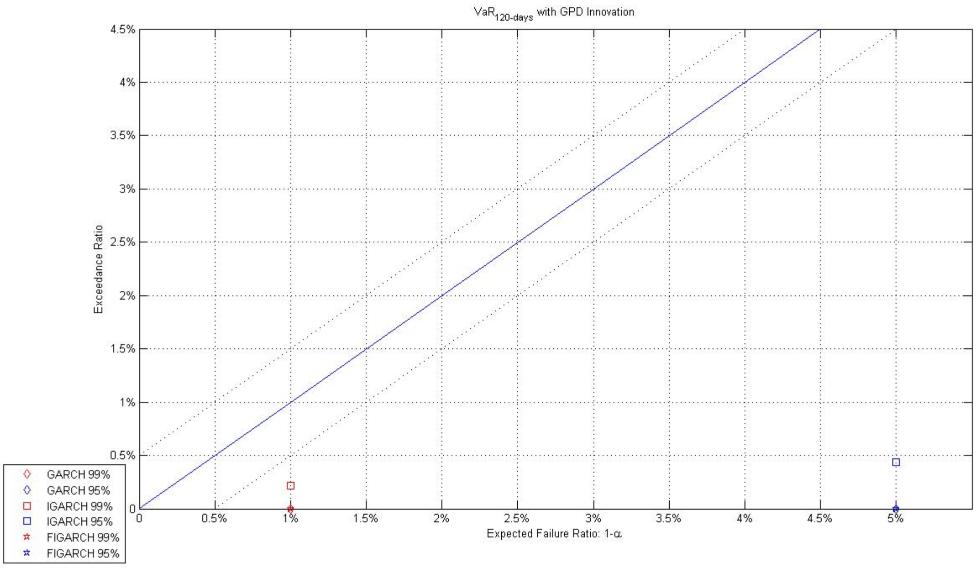

Figure A.12: Exceedance Ratio of In-Sample (1990-2005) 6-Months VaR with GPD Innovation

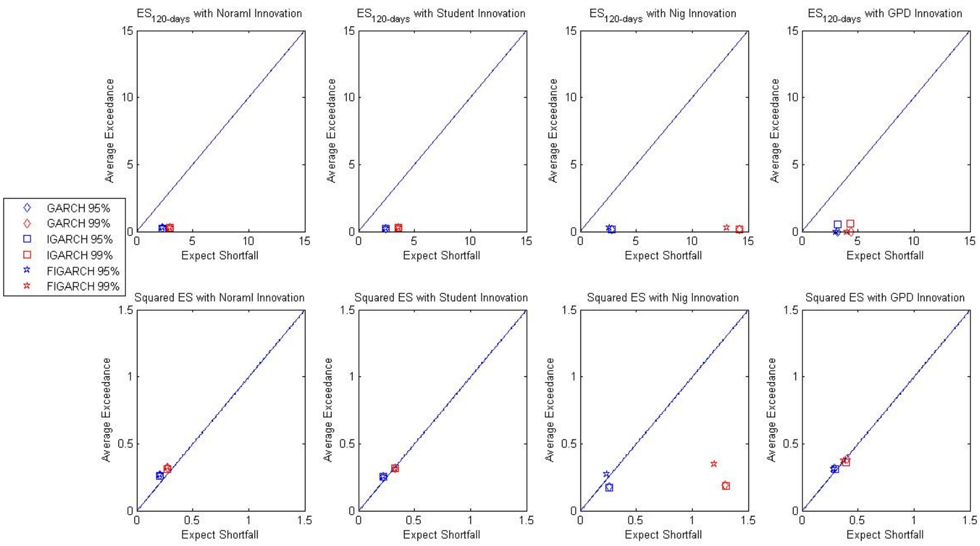

Figure A.13: In-Sample (1990-2005) $ES_{6-months}$ & Square-root estimated 6-Months ES vs. Average Observed 6 months Losses over corresponding VaR

Appendix B. Tables

Distribution	Estimation of Parameters	Confidence Interval
Normal	$u_N = 0.0003$	[0.0000 0.0006]
	$\sigma_N = 0.0100$	[0.0097 0.0102]
Student	$u_t = 0.0004$	[0.0002 0.0007]
	$\sigma_t = 0.0068$	[0.0065 0.0071]
	$\gamma = 3.3799$	[2.9660 3.7938]
NIG	$\alpha_{NIG} = 79.2290$	[79.2222 79.2358]
	$\beta_{NIG} = -3.5299$	[-3.5367 -3.5231]
	$u_{NIG} = 0.0007$	[-0.0062 0.0075]
	$\delta_{NIG} = 0.0079$	[0.0011 0.0148]
GPD	$\alpha_{GPD} = 0.0224$	[-0.0213 0.0661]
	$\beta_{GPD} = 0.0069$	[0.0065 0.0074]
Mix	$\alpha_{Mix} = -0.3929$	[-0.4608 -0.3250]
	$\beta_{Mix} = 0.0296$	[0.0257 0.0341]

Table B.3: Parameters Estimation of AR Model Residuals (Using in-sample data (1990-2005))

Distributions	Parameter	Value	Confidence Interval
Normal	α_0	4.3762e-006	[-0.0007 0.0007]
	α_1	0.9166	[0.9030 0.9302]
	β_1	0.0833	[0.0801 0.0867]
	u_N	-0.0046	[-0.0281 0.7578]
	σ_N	0.7740	[0.0189 0.7911]
Student	α_0	4.9567e-006	[-0.0416 0.0416]
	α_1	0.9544	[-0.5614 2.4702]
	β_1	0.0424	[-0.0352 0.1199]
	u_t	0.0064	[-0.0119 0.0247]
	σ_t	0.5266	[0.5063 0.5477]
	γ	5.2791	[4.3741 6.1840]
NIG	α_0	0.0029	[-0.0317 0.0375]
	α_1	0.5134	[0.4751 0.5499]
	β_1	0.4866	[0.4722 0.5028]
	α_{NIG}	6.4111	[6.4050 6.4172]
	β_{NIG}	-0.2403	[-0.2464 -0.2342]
	u_{NIG}	0.0038	[-0.0023 0.0100]
	γ_{NIG}	0.1045	[0.0984 0.1106]
GPD	α_0	1.0000e-010	1.0e-009*[0.0608 0.1392]
	α_1	0.7992	[-0.4588 2.0572]
	β_1	0.1001	[-0.1052 0.3053]
	α_{GPD}	-0.9475	[-0.8245 1.0232]
	β_{GPD}	17.6894	[15.2356 19.5124]

Table B.4: Parameters Estimation of GARCH Model (Using in-sample data (1990-2005))

Distributions	Parameter	Value	Confidence Interval
Normal	α_0	4.3464e-006	[-0.0393 0.0393]
	α_1	0.9170	[0.1676 1.6665]
	β_1	0.0830	[-0.0632 0.2291]
	u_N	-0.0046	[-0.0280 0.7579]
	σ_N	0.7741	[0.0189 0.7911]
Student	α_0	2.9372e-006	[-0.0014 0.0014]
	α_1	0.9593	[0.8955 1.0232]
	β_1	0.0407	[0.0386 0.0427]
	u_t	0.0071	[-0.0130 0.0272]
	σ_t	0.5819	[0.5598 0.6049]
	γ	5.6877	[4.6594 6.7160]
NIG	α_0	0.0050	[-1.8687 1.8787]
	α_1	0.1772	[-0.0651 0.4195]
	β_1	0.8228	[0.6662 0.9794]
	α_{NIG}	6.4562	[6.4499 6.4625]
	β_{NIG}	-0.2474	[-0.2537 -0.2411]
	u_{NIG}	0.0039	[-0.0024 0.0102]
	γ_{NIG}	0.1037	[0.0974 0.1100]
GPD	α_0	1.0000e-010	1.0e-009*[0.0992 0.1008]
	α_1	0.8497	[0.8058 0.8935]
	β_1	0.1503	[0.1324 0.1683]
	α_{GPD}	-1.0159	[-1.9834 -0.9752]
	β_{GPD}	12.1937	[11.8234 13.0341]

Table B.5: Parameters Estimation of IGARCH Model (Using in-sample data (1990-2005))

Distributions	Parameter	Value	Confidence Interval
Normal	α_0	1.0000e-005	[-0.0051 0.0051]
	α_1	0.7466	[0.7213 0.7720]
	β_1	0.1465	[0.1325 0.1606]
	u_N	-0.0078	[-0.0369 0.9389]
	σ_N	0.9589	[0.0213 0.9801]
Student	α_0	9.0944e-007	[-0.0165 0.0165]
	α_1	0.9460	[0.1681 1.7238]
	β_1	0.0530	[-0.0407 0.1468]
	u_t	0.0089	[-0.0162 0.0340]
	σ_t	0.7377	[0.7105 0.7661]
	γ	6.6577	[5.3018 8.0135]
NIG	α_0	0.3650e-04	[-0.0007 0.0007]
	α_1	0.1034	[-0.0342 0.1758]
	β_1	0.7228	[0.5682 0.8794]
	α_{NIG}	6.4562	[6.4499 6.4625]
	β_{NIG}	-0.2474	[-0.2537 -0.2411]
	u_{NIG}	0.0039	[-0.0024 0.0102]
	γ_{NIG}	0.1037	[0.0974 0.1100]
GPD	α_0	1.0000e-010	[-0.0020 0.0020]
	α_1	0.7404	[0.7318 0.7489]
	β_1	0.1426	[0.1379 0.1473]
	α_{GPD}	-0.0264	[-0.0657 0.6945]
	β_{GPD}	0.7359	[0.0128 0.7797]

Table B.6: Parameters Estimation of FIGARCH Model (Using in-sample data (1990-2005))

(a) 3-Months Ahead VaR Estimation

Model	α	Normal	Student	NIG	GPD
GARCH(1,1)	0.95	0.1133	0.0595	0.1294	1.0953
	0.99	0.1732	0.0986	0.1965	2.0475
IGARCH(1,1,1)	0.95	0.1150	0.0987	0.1297	1.2464
	0.99	0.2057	0.1582	0.1968	2.3678
FIGARCH(1,d,1)	0.95	0.0720	0.0536	0.1296	0.8262
	0.99	0.1117	0.0960	0.1967	1.0428

(b) 3-Months Ahead ES Estimation

Model	α	Normal	Student	NIG	GPD
GARCH(1,1)	0.95	0.1503	0.0835	0.1705	1.8936
	0.99	0.1782	0.1184	0.2304	2.1692
IGARCH(1,1,1)	0.95	0.1542	0.1357	0.1709	2.0025
	0.99	0.2136	0.1910	0.2308	2.5390
FIGARCH(1,d,1)	0.95	0.0962	0.0734	0.1706	0.9669
	0.99	0.1311	0.0871	0.2306	1.2250

Table B.7: Out-of-Sample 3-Months Ahead VaR & ES Estimation (1st Jan. 2006 - 31 Mar. 2006)

(a) 6-Months Ahead VaR Estimation

Model	α	Normal	Student	NIG	GPD
GARCH(1,1)	0.95	0.1394	0.0893	0.1446	1.1149
	0.99	0.2142	0.1541	0.2207	1.2406
IGARCH(1,1,1)	0.95	0.1390	0.1034	0.1448	1.2415
	0.99	0.2171	0.1786	0.2211	1.3591
FIGARCH(1,d,1)	0.95	0.1378	0.1072	0.1451	1.2108
	0.99	0.2237	0.1237	0.2322	1.2924

(b) 6-Months Ahead ES Estimation

Model	α	Normal	Student	NIG	GPD
GARCH(1,1)	0.95	0.1859	0.1300	0.1913	2.0313
	0.99	0.2556	0.1935	0.2596	2.6912
IGARCH(1,1,1)	0.95	0.1876	0.1911	0.1917	2.2130
	0.99	0.2605	0.2315	0.2592	2.8239
FIGARCH(1,d,1)	0.95	0.1565	0.1145	0.2012	2.1524
	0.99	0.2461	0.2046	0.2632	2.7032

Table B.8: Out-of-Sample 6-Months Ahead VaR & ES Estimation (1st Jan. 2006 - 30 June 2006)

(a) VaR at 9th March 2009

Model	α	Normal	Student	NIG	GPD
GARCH(1,1)	0.95	0.2264	0.1060	0.1635	1.2329
	0.99	0.4051	0.2857	0.2379	1.4565
IGARCH(1,1,1)	0.95	0.3360	0.1244	0.1640	1.4072
	0.99	0.5061	0.1941	0.2385	1.5233
FIGARCH(1,d,1)	0.95	0.3759	0.1356	0.1706	1.4752
	0.99	0.5935	0.2953	0.2438	1.4979

(b) ES at 9th March 2009

Model	α	Normal	Student	NIG	GPD
GARCH(1,1)	0.95	0.3365	0.1489	0.1893	1.4332
	0.99	0.4946	0.2545	0.5507	1.6259
IGARCH(1,1,1)	0.95	0.4778	0.2170	0.1928	1.5841
	0.99	0.5552	0.3783	0.5531	1.6765
FIGARCH(1,d,1)	0.95	0.5485	0.2985	0.2025	1.5930
	0.99	0.5827	0.4451	0.6040	1.6221

Table B.9: Out-of-Sample VaR & ES of the Extreme Aggregated Losses*

*(corresponding to the loss holding the portfolio from 1st Jan. To 9th Mar. 2009)

Horizon	Model	Normal	Student	NIG	GPD
1-Month Horizon	GARCH	5.3657e-010	2.3866e-011	7.7329e-009	3.7796e-009
	IGARCH	5.2263e-011	2.1738e-011	6.3105e-010	3.7932e-010
	FIGARCH	5.5323e-010	4.7769e-011	8.4153e-010	3.9951e-010
3-Months Horizon	GARCH	1.2565e-009	3.4237e-010	8.3571e-010	3.3054e-010
	IGARCH	9.4158e-011	3.3919e-010	6.3225e-010	1.1260e-010
	FIGARCH	5.1529e-010	9.5394e-010	8.0521e-010	7.9325e-010
6-Months Horizon	GARCH	1.1388e-009	8.4057e-009	8.9358e-009	7.4795e-009
	IGARCH	1.2229e-010	3.5268e-010	9.2416e-010	5.3853e-010
	FIGARCH	1.1373e-009	6.6063e-009	3.4521e-009	7.5248e-010
1-Year Horizon	GARCH	1.5216e-009	1.2910e-009	3.9208e-009	6.0894e-009
	IGARCH	7.8472e-010	1.0186e-009	8.2418e-010	1.4140e-009
	FIGARCH	1.4652e-009	1.4516e-009	3.0243e-009	1.0105e-009
3-Years Horizon	GARCH	1.7634e-007	2.7759e-007	4.2514e-008	3.7013e-007
	IGARCH	2.5431e-007	3.3589e-007	9.4338e-007	4.1159e-007
	FIGARCH	1.7463e-007	2.7311e-007	2.9783e-007	3.2377e-007
	GARCH	1.2293e-007	1.8351e-007	1.3215e-008	2.6857e-007
5-Years Horizon	IGARCH	1.7963e-007	2.2786e-007	2.0421e-008	3.2783e-007
	FIGARCH	1.2115e-007	1.5631e-007	5.2312e-007	2.2590e-007

Table B.10: Mean Square Error of Volatility Forecast (Out-of-sample from 1st Jan. 2006 to the given horizon)